

SCATTERING THEORY FOR CMV MATRICES: UNIQUENESS, HELSON–SZEGŐ AND STRONG SZEGŐ THEOREMS

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ABSTRACT. We develop a scattering theory for CMV matrices, similar to the Faddeev–Marchenko theory. A necessary and sufficient condition is obtained for the uniqueness of the solution of the inverse scattering problem. We also obtain two sufficient conditions for the uniqueness, which are connected with the Helson–Szegő and the Strong Szegő theorems. The first condition is given in terms of the boundedness of a transformation operator associated to the CMV matrix. In the second case this operator has a determinant. In both cases we characterize Verblunsky parameters of the CMV matrices, corresponding spectral measures and scattering functions.

1. INTRODUCTION

To a given collection of numbers $\{\alpha_n\}_{n \geq 0}$ in the open unit disk \mathbb{D} , called the *Verblunsky coefficients*, and α_{-1} in the unit circle \mathbb{T} , we define the CMV matrix $\mathfrak{A} = \mathfrak{A}_{od}\mathfrak{A}_e$, where

$$\mathfrak{A}_{od} = \begin{bmatrix} -\alpha_{-1} & & & \\ & A_1 & & \\ & & A_3 & \\ & & & \ddots \end{bmatrix}, \quad \mathfrak{A}_e = \begin{bmatrix} A_0 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix},$$

and the A_k 's are the 2×2 unitary matrices

$$A_k = \begin{bmatrix} \overline{\alpha}_k & \rho_k \\ \rho_k & -\alpha_k \end{bmatrix}, \quad \rho_k = \sqrt{1 - |\alpha_k|^2}.$$

Unlike the standard convention [27, p. 265], we do not fix the value $\alpha_{-1} = -1$. Our reasons will become clear later on.

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Note that \mathfrak{A} is a unitary operator on $l^2(\mathbb{Z}_+)$. The initial vector e_0 of the standard basis is cyclic for \mathfrak{A} . Indeed, by the definition for $n = 0, 1, \dots$

$$\begin{aligned}\mathfrak{A}\{e_{2n}\rho_{2n} - e_{2n+1}\bar{\alpha}_{2n}\} &= e_{2n+1}\bar{\alpha}_{2n+1} + e_{2n+2}\rho_{2n+1} \\ \mathfrak{A}^{-1}\{e_{2n+1}\rho_{2n+1} - e_{2n+2}\alpha_{2n+1}\} &= e_{2n+2}\alpha_{2n+2} + e_{2n+3}\rho_{2n+2} \\ \mathfrak{A}^{-1}e_0 &= -\bar{\alpha}_{-1}(e_0\alpha_0 + e_1\rho_0).\end{aligned}\tag{1.1}$$

That is, acting in turn by \mathfrak{A}^{-1} and \mathfrak{A} on e_0 and taking the linear combinations, we can get any vector of the standard basis. CMV matrices were introduced in [8]. More recent surveys on this topic are [27, 28, 21].

1.1. Spectral Characteristics. Since \mathfrak{A} is a unitary operator, then the following function

$$R(z) := \left\langle \frac{\mathfrak{A} + z}{\mathfrak{A} - z} e_0, e_0 \right\rangle = \int_{\mathbb{T}} \frac{t + z}{t - z} \sigma(dt) \tag{1.2}$$

has a nonnegative real part in the unit disk, which yields the integral formula in (1.2). Measure $\sigma = \sigma(\mathfrak{A})$ is called the *spectral measure* of \mathfrak{A} with respect to the cyclic vector e_0 . The standard Lebesgue decomposition is

$$\sigma(dt) = w(t)m(dt) + \sigma_s(dt) \tag{1.3}$$

where $m(dt)$ is the normalized Lebesgue measure, and σ_s is the singular component. We will say that \mathfrak{A} is absolutely continuous if $\sigma_s = 0$. Note that

$$R(0) = \langle e_0, e_0 \rangle = \int_{\mathbb{T}} \sigma(dt) = 1,$$

so σ is a *probability measure*.

We define function ϕ by the equation

$$\phi(z) = \alpha_{-1} \frac{1 - R(z)}{1 + R(z)}, \quad R(z) = \frac{1 - \bar{\alpha}_{-1}\phi(z)}{1 + \bar{\alpha}_{-1}\phi(z)}. \tag{1.4}$$

Then $|\phi| \leq 1$, $\phi(0) = 0$. An important relation is

$$w(t) = \operatorname{Re} R(t) = \frac{1 - |\phi(t)|^2}{|1 + \bar{\alpha}_{-1}\phi(t)|^2} \tag{1.5}$$

a.e. on \mathbb{T} .

The spectral measure σ is uniquely determined from the CMV matrix \mathfrak{A} by (1.2). Conversely, by the first formula in (1.4), the measure σ uniquely defines $\bar{\alpha}_{-1}\phi$. Hence, to recover ϕ (and by that α_n), we need to know α_{-1} . Therefore, the pair $\{\sigma, \alpha_{-1}\}$, not just σ , determines uniquely the CMV matrix \mathfrak{A} . That is why we consider the pair $\{\sigma, \alpha_{-1}\}$ as the spectral data.

The one-to-one correspondences

$$\mathfrak{A} \longleftrightarrow \{\sigma, \alpha_{-1}\} \longleftrightarrow \{R, \alpha_{-1}\} \longleftrightarrow \{\phi, \alpha_{-1}\}$$

are studied in the theory of orthogonal polynomials on the unit circle (OPUC) [27] and in the Schur analysis [26].

1.2. Direct scattering. By definition, the matrix \mathfrak{A} is in the *Szegő class*, $\mathfrak{A} \in \mathbf{Sz}$, if $\sum |\alpha_k|^2 < \infty$. It is known that $\mathfrak{A} \in \mathbf{Sz}$ if and only if the spectral measure σ is of the form

$$\mathfrak{A} \in \mathbf{Sz} \Leftrightarrow \log w \in L^1, \quad (1.6)$$

see [27, Theorem 2.3.1]. The standard fact from the theory of Hardy classes reads that assumption (1.6) yields

$$w(t) = |D(t)|^2 \quad (1.7)$$

a.e., where D is a boundary value of an outer H^2 function, $D(0) > 0$. D is known as the *Szegő function*. By the Szegő theorem

$$D(0) = \prod_{k=0}^{\infty} \rho_k. \quad (1.8)$$

It follows from (1.5) that

$$\mathfrak{A} \in \mathbf{Sz} \Leftrightarrow \log(1 - |\phi|^2) \in L^1,$$

so an outer function ψ , which satisfies

$$|\psi(t)|^2 + |\phi(t)|^2 = 1, \quad \psi(0) > 0, \quad (1.9)$$

is well defined, uniquely determined by ϕ . By (1.5)

$$w(t) = \left| \frac{\psi(t)}{1 + \bar{\alpha}_{-1}\phi(t)} \right|^2 \quad (1.10)$$

a.e. Hence D is of the form

$$D(z) = \frac{\psi(z)}{1 + \bar{\alpha}_{-1}\phi(z)}, \quad \psi(0) = D(0) = \prod_{k=0}^{\infty} \rho_k. \quad (1.11)$$

Definition 1.1. The *scattering function* of \mathfrak{A} is defined as

$$s(t) = -\bar{\alpha}_{-1} \frac{D(t)}{\overline{D(t)}} = -\frac{\psi(t)}{\psi(t)} \frac{\bar{\alpha}_{-1} + \bar{\phi}(t)}{1 + \bar{\alpha}_{-1}\phi(t)}, \quad t \in \mathbb{T}. \quad (1.12)$$

Note that $|s(t)| = 1$ a.e. on \mathbb{T} .

In the Faddeev–Marchenko theory the scattering function appears as a coefficient in the leading term of certain asymptotics. In our context we have

Theorem 1.2. *Let $\mathfrak{A} \in \mathbf{Sz}$. Then there exists a unique generalized eigenvector $\Psi(t) = \{\Psi_n(t)\}_{n=0}^{\infty}$ such that*

$$\begin{bmatrix} \Psi_0(t) & \Psi_1(t) & \dots \end{bmatrix} \mathfrak{A} = t \begin{bmatrix} \Psi_0(t) & \Psi_1(t) & \dots \end{bmatrix}, \quad t \in \mathbb{T}, \quad (1.13)$$

and the following asymptotics holds in L^2 -norm

$$\Psi_{2n}(t) = t^n + o(1), \quad \Psi_{2n+1}(t) = \overline{s(t)}t^{-n-1} + o(1), \quad n \rightarrow \infty. \quad (1.14)$$

Theorem 1.2 is a restatement of the classical Szegő theorem on the asymptotic behavior of OPUC [27, Theorem 2.4.1], since we can choose

$$\Psi_{2n}(t) = \overline{D(t)}t^{-n}p_{2n}(t), \quad \Psi_{2n+1}(t) = -\alpha_{-1}\overline{D(t)}t^n\overline{p_{2n+1}(t)}$$

as a solution of (1.13), where p_n are orthonormal polynomials with respect to σ (cf. [27, Lemma 4.3.14]).

1.3. Main Objectives and Results. The main objective of this paper is solving the inverse scattering problem (the heart of the Faddeev–Marchenko theory [19, 20, 10]), i.e., reconstructing the CMV matrix \mathfrak{A} from its scattering function s . In general, the solution of this inverse problem is not unique. In particular, s does not contain any information about the (possible) singular measure. Even in the class of absolutely continuous measures the correspondence $\mathfrak{A} \mapsto s$ is not one to one (see Examples 3.4 and 7.13). In this paper we show that the uniqueness in the inverse scattering is equivalent to the *Arov regularity* (Definition 2.4) of the function ϕ , see Theorem 3.1 below.

We also consider two interesting subclasses of the uniqueness class, namely, Helson–Szegő and the Strong Szegő. The first class is exactly the one for which a certain *transformation operator*¹ is invertible. We obtain a complete description of the corresponding spectral measures and the scattering functions in Section 6. The second class is the one for which the transformation operators have a determinant. For this class a complete description is given to the Verblunsky coefficients, the spectral measures and the scattering functions in Section 7.

This paper is the result of a substantial revision of the manuscript [14].

2. ADAMYAN–AROV–KREIN THEORY

We begin with the following

Definition 2.1. Pairs (ϕ, ψ) with properties ϕ, ψ are in H^∞ , $\phi(0) = 0$, ψ is an outer function, $\psi(0) > 0$, and $|\phi|^2 + |\psi|^2 = 1$ are called *γ -generating*.

Recall that such pairs appear in spectral analysis of CMV matrices (see Introduction).

Proposition 2.2. *To every γ -generating pair (ϕ, ψ) one can associate the family of functions (compare to (1.12))*

$$s_{\mathcal{E}} = -\frac{\psi}{\overline{\psi}} \frac{\mathcal{E} + \overline{\phi}}{1 + \mathcal{E}\phi}, \quad \mathcal{E} \in H^\infty, \quad \|\mathcal{E}\|_\infty \leq 1. \quad (2.1)$$

All the functions $s_{\mathcal{E}}$ belong to the unit ball of L^∞ . Moreover, all functions in formula (2.1) have the same negative part of the Fourier series.

¹A classical monograph on the subject is [19], where transformation operators are extensively used in spectral and scattering theory for Schrödinger operator. Historical remarks are also given there in the introduction.

Proof. The first assertion follows from the relation

$$1 - |s_{\mathcal{E}}|^2 = \frac{(1 - |\mathcal{E}|^2)(1 - |\phi|^2)}{|1 + \mathcal{E}\phi|^2}.$$

Let s_0 correspond to $\mathcal{E} = 0$, then

$$s_{\mathcal{E}} - s_0 = -\frac{\psi}{\bar{\psi}} \frac{\mathcal{E} + \bar{\phi}}{1 + \mathcal{E}\phi} + \frac{\psi}{\bar{\psi}} \frac{\bar{\phi}}{\phi} = -\frac{\psi^2 \mathcal{E}}{1 + \mathcal{E}\phi} \in H^{\infty}. \quad (2.2)$$

□

The following observation will be helpful later on. For each γ -generating pair (ϕ, ψ) and any Schur class function \mathcal{E} the function

$$D_{\mathcal{E}}(z) := \frac{\psi(z)}{1 + \mathcal{E}(z)\phi(z)} \quad (2.3)$$

is an outer function from H^2 . Indeed, $D_{\mathcal{E}}$ is the outer function (as a ratio of outer functions) from the Smirnov class, and

$$|D_{\mathcal{E}}(t)|^2 = \frac{1 - |\phi(t)|^2}{|1 + \mathcal{E}(t)\phi(t)|^2} \leq \frac{1 - |\mathcal{E}(t)\phi(t)|^2}{|1 + \mathcal{E}(t)\phi(t)|^2} = \operatorname{Re} \frac{1 - \mathcal{E}(t)\phi(t)}{1 + \mathcal{E}(t)\phi(t)}.$$

The right hand side is the boundary value of the Poisson integral of a finite positive measure, and so belongs to $L^1(\mathbb{T})$.

The AAK Theory deals with the following Nehari problem [1, 2, 3, 11].

Problem 2.3 (Nehari). Given function $h \in L^{\infty}$, $\|h\|_{\infty} \leq 1$, describe collection $\mathcal{N}(h)$ of all functions

$$\mathcal{N}(h) = \{f \in L^{\infty} : \|f\|_{\infty} \leq 1, f - h \in H^{\infty}\},$$

that is, the collection of functions $f \in L^{\infty}$ with the same Fourier coefficients with negative indices as h .

The Nehari problem is indeterminate (determinate) if it has infinitely many solutions (a unique solution). It follows from Proposition 2.2 that s (1.12) is a unimodular solution of indeterminate Nehari problem.

By Proposition 2.2 for every γ -generating pair (ϕ, ψ) the family $\{s_{\mathcal{E}}\}$ (2.1) solves a certain Nehari problem, generated by, e.g., s_0 . However, formula (2.1) may not produce *all the functions* from the unit ball of L^{∞} with *this* negative part of the Fourier series.

Definition 2.4. A γ -generating pair (ϕ, ψ) , or simply a function ϕ , are called *Arov-regular* (see [4]) if formula (2.1) produces *all the functions* from the unit ball of L^{∞} with a certain negative part of the Fourier series.

Definition 2.5. We say that a CMV matrix of the Szegő class is regular, if the associated function ϕ (1.4) is Arov-regular.

An important result is proved in [2, Remark 4.1].

Theorem 2.6 (AAK). *If ϕ is Arov-regular, then for every Schur class function \mathcal{E} the measure $\sigma_{\mathcal{E}}$*

$$\frac{1 - \mathcal{E}(z)\phi(z)}{1 + \mathcal{E}(z)\phi(z)} = \int_{\mathbb{T}} \frac{t + z}{t - z} \sigma_{\mathcal{E}}(dt) \quad (2.4)$$

is absolutely continuous.

For $h \in L^\infty$, $\|h\|_\infty \leq 1$, we define a Hankel operator $\mathcal{H} : H^2 \rightarrow H_-^2$ as

$$\mathcal{H} = \mathcal{H}_h = P_- h|H^2,$$

h is called a *symbol* of \mathcal{H} . Note that

$$\|\mathcal{H}\| \leq \|h\|_\infty \leq 1,$$

and the adjoint operator $\mathcal{H}^* : H_-^2 \rightarrow H^2$ is $\mathcal{H}^* = P_+ \bar{h}|H_-^2$, P_+ (P_-) is the standard projection from L^2 onto H^2 (H_-^2). For $\|f\|_\infty \leq 1$, $\mathcal{H}_f = \mathcal{H}$ if and only if $f \in \mathcal{N}(\mathcal{H})$.

A Hankel operator \mathcal{H}_h is called *indeterminate*, if it has many symbols f with $\|f\|_\infty \leq 1$.

Theorem 2.7 (Adamyman–Arov–Krein). *The Nehari problem is indeterminate if and only if*

$$\mathbf{1} \in (I - \mathcal{H}^* \mathcal{H})^{1/2} H^2. \quad (2.5)$$

In this case the set $\mathcal{N}(\mathcal{H})$ is of the form

$$\mathcal{N}(\mathcal{H}) = \{f_{\mathcal{E}} = -\frac{\psi_{\mathcal{H}}}{\bar{\psi}_{\mathcal{H}}} \frac{\mathcal{E} + \bar{\phi}_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}} : \mathcal{E} \in H^\infty, \|\mathcal{E}\|_\infty \leq 1\}, \quad (2.6)$$

where $(\phi_{\mathcal{H}}, \psi_{\mathcal{H}})$ is a uniquely determined Arov-regular pair, $\psi_{\mathcal{H}}(0) > 0$.

The next theorem gives sufficient conditions for regularity of ϕ . The second condition is known (see, e.g., [4, 25]). For a weaker condition on $|\psi|$, which ensures regularity of ϕ , see [29].

Theorem 2.8. *ϕ is Arov-regular as soon as one of the following conditions holds*

- (1) σ_τ (2.4) *is absolutely continuous for some unimodular constant τ , and $(1 + \tau\phi)\psi^{-1} \in H^2$;*
- (2) $\psi^{-1} \in H^2$.

Proof. (1). We consider a unimodular function

$$s = -\frac{\psi}{\bar{\psi}} \frac{\tau + \bar{\phi}}{1 + \tau\phi} = -\tau \frac{\psi}{\bar{\psi}} \frac{1 + \bar{\tau}\bar{\phi}}{1 + \tau\phi}. \quad (2.7)$$

We associate an indeterminate Nehari problem to s with the Hankel operator $\mathcal{H} = \mathcal{H}_s$. By Theorem 2.7 s admits the representation

$$s = -\frac{\psi_{\mathcal{H}}}{\bar{\psi}_{\mathcal{H}}} \frac{\mathcal{E} + \bar{\phi}_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}} \quad (2.8)$$

with the Arov-regular pair $(\phi_{\mathcal{H}}, \psi_{\mathcal{H}})$ and the inner function \mathcal{E} , so we can write

$$s = -\mathcal{E} \frac{\psi_{\mathcal{H}}}{\psi} \frac{1 + \overline{\mathcal{E}\phi_{\mathcal{H}}}}{1 + \mathcal{E}\phi_{\mathcal{H}}}.$$

Combining (2.7) and (2.8), we get

$$G := \mathcal{E} \frac{1 + \tau\phi}{\psi} \frac{\psi_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}} = \tau \frac{1 + \tau\phi}{\psi} \frac{\psi_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}}. \quad (2.9)$$

It was mentioned above (see (2.3)) that $\psi_{\mathcal{H}}(1 + \mathcal{E}\phi_{\mathcal{H}})^{-1} \in H^2$, so, due to the assumption, $G \in H^1$. At the same time $\overline{G} \in H^1$, so G is a constant function. Since \mathcal{E} is the inner part of G , we have $\mathcal{E} = \text{const}$. Using the normalization $\psi(0) > 0, \psi_{\mathcal{H}}(0) > 0$, we get $\mathcal{E} = \tau$ and $\tau G > 0$. Next, by (2.9) Next

$$\tau G \frac{\psi}{1 + \tau\phi} = \frac{\psi_{\mathcal{H}}}{1 + \tau\phi_{\mathcal{H}}}.$$

so, in particular,

$$(\tau G)^2 \left| \frac{\psi}{1 + \tau\phi} \right|^2 = \left| \frac{\psi_{\mathcal{H}}}{1 + \tau\phi_{\mathcal{H}}} \right|^2.$$

In other words,

$$(\tau G)^2 \operatorname{Re} \frac{1 - \tau\phi}{1 + \tau\phi} = \operatorname{Re} \frac{1 - \tau\phi_{\mathcal{H}}}{1 + \tau\phi_{\mathcal{H}}}$$

almost everywhere on the unit circle.

By the assumption σ_{τ} is absolutely continuous, and by Theorem 2.6 $\sigma_{\tau, \mathcal{H}}$ is absolutely continuous. Since $\phi(0) = \phi_{\mathcal{H}}(0) = 0$, σ_{τ} and $\sigma_{\tau, \mathcal{H}}$ are probability measures. Hence, $\tau G = 1$,

$$\frac{1 - \tau\phi}{1 + \tau\phi} = \frac{1 - \tau\phi_{\mathcal{H}}}{1 + \tau\phi_{\mathcal{H}}}.$$

Therefore $\phi = \phi_{\mathcal{H}}$, as claimed.

(2). Let us show first that σ is absolutely continuous. Indeed, by (1.11)

$$\frac{1}{1 + \overline{\alpha}_{-1}\phi(z)} = \frac{D(z)}{\psi(z)} \in H^1 \Rightarrow R(z) = \frac{1 - \overline{\alpha}_{-1}\phi(z)}{1 + \overline{\alpha}_{-1}\phi(z)} \in H^1.$$

By the Fihthengoltz theorem

$$R(z) = \int_{\mathbb{T}} \frac{R(t)}{1 - \overline{t}z} m(dt),$$

and

$$1 = R(0) = \int_{\mathbb{T}} R(t) m(dt) = \int_{\mathbb{T}} \operatorname{Re} R(t) m(dt) = \int_{\mathbb{T}} w(t) m(dt),$$

so $\sigma_s = 0$, as claimed. Next, by the assumption

$$\frac{1}{D(z)} = \frac{1 + \overline{\alpha}_{-1}\phi(z)}{\psi(z)} \in H^2,$$

and the second statement of the theorem follows from the first one. \square

Definition 2.9. If ϕ is Arov-regular and \mathcal{E} is a constant function, $|\mathcal{E}| = 1$, then the function

$$s_{\mathcal{E}} = -\frac{\psi}{\bar{\psi}} \frac{\mathcal{E} + \bar{\phi}}{1 + \mathcal{E}\phi}$$

is called *canonical*. Such $s_{\mathcal{E}}$ is also called a *canonical symbol* of the associated Hankel operator \mathcal{H} .

Proposition 2.10. [1, 2, 25, 17] *Let s be a unimodular function on \mathbb{T} . Then the following are equivalent*

- (1) *s is canonical,*
- (2) *$P_+ s|H_+^2$ is dense in H_+^2 ,*
 $P_+ t s|H_+^2$ is not dense in H_+^2 (the space is of codimension one),
- (3) *$\bar{s}h_+ = h_-$ has only the trivial solution,*
 $\bar{s}h_+ = th_-$ has a nontrivial solution (the space of solutions is of dimension one), $h_{\pm} \in H_{\pm}^2$.

As a simple consequence of Proposition 2.10 we have

Proposition 2.11. *Let s be canonical, and $N \neq 0$ an integer. Then st^N is non-canonical.*

Proof. Assume that both s and st^N are canonical. Then without loss of generality we may assume that $N > 0$. By the second condition (3) the equation $\bar{s}h_+ = th_-$ has a nontrivial solution. Hence $\overline{st^N}h_+ = t^{1-N}h_- \in H_-^2$ also has a nontrivial solution, which means that the first condition in (3) fails for st^N . So st^N is non-canonical, which is a contradiction. \square

3. UNIQUENESS IN THE INVERSE SCATTERING

We are interested in the following questions: given a unimodular solution s of an indeterminate Nehari problem, does there exist CMV matrix \mathfrak{A} with this scattering function? Is such \mathfrak{A} unique? The main result of the section gives complete answers on these questions.

Theorem 3.1.

- (1) *Each regular CMV matrix \mathfrak{A} has absolutely continuous spectral measure $\sigma(\mathfrak{A})$, and its scattering function s is canonical.*
- (2) *Let s be a canonical solution of an indeterminate Nehari problem, then there exists a unique absolutely continuous CMV matrix \mathfrak{A} of Szegő class, whose scattering function is s , moreover \mathfrak{A} is regular.*
- (3) *Let s be a non-canonical unimodular solution of an indeterminate Nehari problem, then there exist infinitely many absolutely continuous CMV matrices \mathfrak{A} with scattering function s .*

Proof. (1). Let \mathfrak{A} be regular, so ϕ in (1.12) is Arov-regular. By Theorem 2.6, $\sigma(\mathfrak{A})$ is absolutely continuous. By definition 2.9, function s , defined by (1.12) with $\mathcal{E} = \bar{\alpha}_{-1}$, is canonical.

(2). Since s is canonical, we have

$$s = -\frac{\psi_{\mathcal{H}}}{\overline{\psi_{\mathcal{H}}}} \frac{\mathcal{E} + \overline{\phi_{\mathcal{H}}}}{1 + \mathcal{E}\phi_{\mathcal{H}}}, \quad (3.1)$$

where \mathcal{E} is a unimodular constant. Therefore, a solution of the inverse scattering problem can be chosen as

$$\overline{\alpha}_{-1} := \mathcal{E}, \quad D(z) := \frac{\psi_{\mathcal{H}}(z)}{1 + \mathcal{E}\phi_{\mathcal{H}}(z)}, \quad \sigma(dt) := \left| \frac{\psi_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}} \right|^2 m(dt).$$

Since

$$R(z) = \int_{\mathbb{T}} \frac{t+z}{t-z} \sigma(dt) = \frac{1 - \overline{\alpha}_{-1}\phi_{\mathcal{H}}(z)}{1 + \overline{\alpha}_{-1}\phi_{\mathcal{H}}(z)},$$

the associated to σ function $\phi = \phi_{\mathcal{H}}$, so ϕ is regular, as needed.

Assume that there are two absolutely continuous CMV matrices \mathfrak{A} and \mathfrak{A}' of Szegő class with the scattering function s . The corresponding spectral measures are $\sigma = |D|^2 m$ and $\sigma' = |D'|^2 m$,

$$\int_{\mathbb{T}} |D|^2 m(dt) = \int_{\mathbb{T}} |D'|^2 m(dt) = 1, \quad D(0) > 0, \quad D'(0) > 0. \quad (3.2)$$

Then we have

$$s(t) = -\overline{\alpha}_{-1} \frac{D(t)}{\overline{D(t)}} = -\overline{\alpha}'_{-1} \frac{D'(t)}{\overline{D'(t)}} \quad (3.3)$$

and

$$-\overline{\alpha}_{-1} D(t) \overline{s(t)} = \overline{D(t)} \quad -\overline{\alpha}'_{-1} D'(t) \overline{s(t)} = \overline{D'(t)}.$$

There exist two real nonzero constants α and α' such that

$$\alpha D(0) + \alpha' D'(0) = 0,$$

Then

$$h_- = \overline{\alpha D + \alpha' D'} \in H_-^2, \quad h_+ = -\overline{\alpha}_{-1} \alpha D - \overline{\alpha}'_{-1} \alpha' D' \in H_+^2$$

is a solution of $\overline{s}h_+ = h_-$. Since s is canonical, by Proposition 2.10, (3), this is a trivial solution. In other words,

$$\alpha D + \alpha' D' = 0$$

identically. In view of (3.2), this yields $D = D'$. The uniqueness follows.

(3). If s is a non-canonical unimodular solution of an indeterminate Nehari problem, then in (3.1) \mathcal{E} is a non-constant inner function, and (3.1) can be rephrased as

$$s = \mathcal{E} \frac{\psi_{\mathcal{H}}}{\overline{\psi_{\mathcal{H}}}} \frac{1 + \overline{\mathcal{E}\phi_{\mathcal{H}}}}{1 + \mathcal{E}\phi_{\mathcal{H}}} = \overline{\tau} \frac{1 - \tau \mathcal{E}}{1 - \overline{\tau} \overline{\mathcal{E}}} \frac{\psi_{\mathcal{H}}}{\overline{\psi_{\mathcal{H}}}} \frac{1 + \overline{\mathcal{E}\phi_{\mathcal{H}}}}{1 + \mathcal{E}\phi_{\mathcal{H}}}, \quad \forall \tau \in \mathbb{T}.$$

Therefore, we get infinitely many solutions of the inverse scattering problem

$$\overline{\alpha}_{-1} = -\frac{\overline{\tau} - \mathcal{E}(0)}{1 - \overline{\tau} \overline{\mathcal{E}(0)}}, \quad D(z) = k_{\tau} \frac{|1 - \tau \mathcal{E}(0)|}{1 - \tau \mathcal{E}(0)} \frac{(1 - \tau \mathcal{E}(z)) \psi_{\mathcal{H}}(z)}{1 + \mathcal{E}(z) \phi_{\mathcal{H}}(z)} \in H^2,$$

where $k_\tau > 0$ is chosen to make $\int_{\mathbb{T}} |D(t)|^2 m(dt) = 1$. It is verified by a straightforward computation that indeed $\bar{\alpha}_{-1}$ and $|D|$ are different for different τ . \square

Corollary 3.2.

- (1) *Let \mathfrak{A} be a regular CMV matrix, let \mathfrak{A}_1 be an absolutely continuous CMV matrix of the Szegő class. If they have the same scattering function s then $\mathfrak{A}_1 = \mathfrak{A}$.*
- (2) *Let \mathfrak{A} be a non-regular absolutely continuous CMV matrix of the Szegő class with the scattering function s . Then there exist infinitely many absolutely continuous CMV matrices of the Szegő class with the same scattering function.*

Remark 3.3. As we saw earlier, for every CMV matrix of Szegő class, its scattering function is a unimodular solution of an indeterminate Nehari problem. As a byproduct of this section, we have shown that every unimodular solution of an indeterminate Nehari problem is the scattering function of an absolutely continuous CMV matrix \mathfrak{A} .

We complete with a simple example, when the solution of the inverse scattering problem is not unique.

Example 3.4. Let

$$P(z) = \prod_{j=1}^N (z - t_j), \quad t_j \in \mathbb{T}$$

be a monic polynomial of degree N with all zeros on \mathbb{T} . For the measure

$$\sigma(dt) = w(t)m(dt), \quad w(t) := c|P(t)|^2 = c \prod_{j=1}^N |t - t_j|^2, \quad c > 0,$$

the Szegő function $D = \sqrt{c}P/P(0)$, and the scattering function is

$$s(t) = -\bar{\alpha}_{-1} \frac{D(t)}{\overline{D(t)}} = -\bar{\alpha}_{-1} \overline{P(0)} t^N.$$

Thus for any two polynomials P_1, P_2 with $P_1(0) = P_2(0)$ we have $s_1 = s_2$, and there is no uniqueness in the inverse scattering even for $\alpha_{-1} = -1$. Note that s is not canonical.

In the case $N = 1$ we have $s = \bar{\alpha}_{-1} \bar{t}_1 t$, and again there is no uniqueness.

4. SCHUR ALGORITHM

It is convenient to deal with two sequences $\{f_n\}_{n \geq 0}$ and $\{\phi_n\}_{n \geq 0}$ given for $n = 0, 1, \dots$ by

$$\begin{aligned} f_{n+1}(z) &= \frac{f_n(z) - f_n(0)}{z(1 - \overline{f_n(z)} \overline{f_n(0)})}, \quad z f_0(z) = \phi_0(z) = \phi(z); \\ \phi_n(z) &= z f_n(z) \end{aligned} \tag{4.1}$$

from the Schur class. By the Geronimus theorem

$$f_n(0) = a_n := -\alpha_{-1} \alpha_n, \quad n = 0, 1, \dots \quad (4.2)$$

If ϕ is a Szegő function, then all the functions ϕ_n (4.1) are also Szegő functions. So, we can define a sequence of γ -generating pairs (ϕ_n, ψ_n) . It is easy to see that $\{\psi_n\}$ satisfies

$$\psi_{k+1} = \psi_k \frac{\rho_k}{1 - \bar{a}_k f_k}, \quad \psi_n = \psi \prod_{k=0}^{n-1} \frac{\rho_k}{1 - \bar{a}_k f_k}, \quad \psi = \psi_0. \quad (4.3)$$

Indeed, for $t \in \mathbb{T}$

$$|\psi_{k+1}(t)|^2 = 1 - |f_{k+1}(t)|^2 = \frac{(1 - |f_k(t)|^2)(1 - |\alpha_k|^2)}{|1 - \bar{a}_k f_k(t)|^2} = \frac{|\psi_k(t)|^2 \rho_k^2}{|1 - \bar{a}_k f_k(t)|^2}.$$

It is also clear from (1.11) and (4.2) that

$$\psi_n(0) = \psi(0) \prod_{k=0}^{n-1} \rho_k^{-1} = \prod_{k=n}^{\infty} \rho_k. \quad (4.4)$$

Lemma 4.1. *Recurrences (4.1) and (4.3) can be put into the form*

$$\begin{bmatrix} \frac{1}{\psi_n} & \frac{\bar{\phi}_n}{\psi_n} \\ \frac{\phi_n}{\psi_n} & \frac{1}{\psi_n} \end{bmatrix} = \begin{bmatrix} \bar{t} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\psi_{n+1}} & \frac{\bar{\phi}_{n+1}}{\psi_{n+1}} \\ \frac{\phi_{n+1}}{\psi_{n+1}} & \frac{1}{\psi_{n+1}} \end{bmatrix} \begin{bmatrix} t & \bar{a}_n \\ a_n t & 1 \end{bmatrix} \frac{1}{\rho_n}. \quad (4.5)$$

Proof. By (4.1),

$$(1 - \bar{a}_n f_n) \phi_{n+1} = f_n - a_n,$$

and

$$(1 - \bar{a}_n f_n)(1 + \bar{a}_n \phi_{n+1}) = 1 - |a_n|^2 = \rho_n^2.$$

Therefore,

$$\frac{\rho_n^2}{1 - \bar{a}_n f_n} = 1 + \bar{a}_n \phi_{n+1}.$$

Next, by (4.3),

$$\frac{1}{\psi_n} = \frac{1}{\psi_{n+1}} \frac{\rho_n}{1 - \bar{a}_n f_n} = \frac{1 + \bar{a}_n \phi_{n+1}}{\psi_{n+1} \rho_n} = \left(\frac{\phi_{n+1}}{\psi_{n+1}} \bar{a}_n + \frac{1}{\psi_{n+1}} \right) \frac{1}{\rho_n},$$

which is (2, 2) entry of (4.5).

Similarly, by (4.1),

$$(1 - \bar{a}_n f_n)(\phi_{n+1} + a_n) = \bar{t} \rho_n^2 \phi_n,$$

and

$$\frac{\rho_n^2 \phi_n}{1 - \bar{a}_n f_n} = t(\phi_{n+1} + a_n).$$

Therefore, by (4.3),

$$\frac{\phi_n}{\psi_n} = \frac{1}{\psi_{n+1}} \frac{\rho_n \phi_n}{1 - \bar{a}_n f_n} = \frac{t(\phi_{n+1} + a_n)}{\psi_{n+1} \rho_n} = t \left(\frac{\phi_{n+1}}{\psi_{n+1}} + \frac{1}{\psi_{n+1}} a_n \right) \frac{1}{\rho_n},$$

which is $(2, 1)$ entry of (4.5). \square

Repeatedly applying (4.5) we get for $n > j$

$$\begin{bmatrix} \frac{1}{\psi_j} & \frac{\bar{\phi}_j}{\psi_j} \\ \frac{\phi_j}{\psi_j} & \frac{1}{\psi_j} \end{bmatrix} = \begin{bmatrix} t^{(n-j)} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\psi_n} & \frac{\bar{\phi}_n}{\psi_n} \\ \frac{\phi_n}{\psi_n} & \frac{1}{\psi_n} \end{bmatrix} \left(\prod_{k=j}^{n-1} \begin{bmatrix} t & \bar{a}_k \\ a_k t & 1 \end{bmatrix} \frac{1}{\rho_k} \right). \quad (4.6)$$

We define

$$\begin{bmatrix} \mathcal{P}_j^j \\ \mathcal{Q}_j^j \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (4.7)$$

and for $n > j$

$$\begin{bmatrix} \mathcal{P}_n^j(z) \\ \mathcal{Q}_n^j(z) \end{bmatrix} = \left(\prod_{k=j}^{n-1} \begin{bmatrix} z & \bar{a}_k \\ a_k z & 1 \end{bmatrix} \frac{1}{\rho_k} \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (4.8)$$

Note that \mathcal{P}_n^j and \mathcal{Q}_n^j are polynomials,

$$\deg \mathcal{P}_n^j \leq n - j - 1, \quad \deg \mathcal{Q}_n^j \leq n - j - 1, \quad \mathcal{Q}_n^j(0) = \prod_{k=j}^{n-1} \rho_k^{-1} > 0. \quad (4.9)$$

It is easily seen from (4.8) that

$$\begin{bmatrix} \mathcal{P}_n^0 & \mathcal{P}_n^j \\ \mathcal{Q}_n^0 & \mathcal{Q}_n^j \end{bmatrix} = \left(\prod_{k=j}^{n-1} \begin{bmatrix} t & \bar{a}_k \\ a_k t & 1 \end{bmatrix} \frac{1}{\rho_k} \right) \begin{bmatrix} \mathcal{P}_j^0 & 0 \\ \mathcal{Q}_j^0 & 1 \end{bmatrix}.$$

Taking determinants we come to

$$\mathcal{P}_n^0(z) \mathcal{Q}_n^j(z) - \mathcal{Q}_n^0(z) \mathcal{P}_n^j(z) = z^{n-j} \mathcal{P}_j^0(z). \quad (4.10)$$

From (4.6) and (4.8) we have

$$\frac{\bar{\phi}_j}{\psi_j} = t^{j-n} \frac{\mathcal{P}_n^j + \bar{\phi}_n \mathcal{Q}_n^j}{\bar{\psi}_n}, \quad \frac{1}{\psi_j} = \frac{\mathcal{P}_n^j \phi_n + \mathcal{Q}_n^j}{\psi_n}. \quad (4.11)$$

Remark 4.2. Matrix products (4.8) arise in the Szegő recurrences for OPUC (see [27, formula (1.5.35)]).

We also define

$$\mathcal{E}_n^j = \frac{\mathcal{P}_n^j}{\mathcal{Q}_n^j}, \quad n \geq j. \quad (4.12)$$

It is clear from (4.8) that \mathcal{E}_n^j can be defined recursively as

$$\mathcal{E}_j^j = 0, \quad \mathcal{E}_{n+1}^j = \frac{t \mathcal{E}_n^j + \bar{a}_n}{1 + a_n t \mathcal{E}_n^j}, \quad n \geq j, \quad (4.13)$$

so $\|\mathcal{E}_n^j\|_\infty < 1$ for $n \geq j$.

Remark 4.3. Using those notations we can rewrite (4.10) as

$$\mathcal{E}_n^0(z) - \mathcal{E}_n^j(z) = \frac{\mathcal{P}_n^0(z)\mathcal{Q}_n^j(z) - \mathcal{Q}_n^0(z)\mathcal{P}_n^j(z)}{\mathcal{Q}_n^0(z)\mathcal{Q}_n^j(z)} = \frac{z^{n-j}\mathcal{P}_j^0(z)}{\mathcal{Q}_n^0(z)\mathcal{Q}_n^j(z)}, \quad (4.14)$$

which implies, in view of (4.9), that the difference

$$\mathcal{E}_n^0(z) - \mathcal{E}_n^j(z) \quad (4.15)$$

vanishes at the origin with order of at least $n - j$.

The second equality in (4.11) also can be rewritten as

$$\frac{\psi_n}{\psi_j} = \mathcal{P}_n^j\phi_n + \mathcal{Q}_n^j = \mathcal{Q}_n^j(1 + \mathcal{E}_n^j\phi_n).$$

Hence,

$$\frac{\psi_n}{\psi_j} \frac{1}{1 + \mathcal{E}_n^j\phi_n} = \mathcal{Q}_n^j \quad (4.16)$$

and, therefore, is a polynomial of degree at most $n - j - 1$.

Lemma 4.4. Let $s_k := -\frac{\psi_k}{\psi_k} \bar{\phi}_k$. Then, for $n \geq j$

$$s_j = -\bar{t}^{(n-j)} \frac{\psi_n}{\psi_n} \frac{\mathcal{E}_n^j + \bar{\phi}_n}{1 + \mathcal{E}_n^j\phi_n} \quad (4.17)$$

and (cf. (2.2))

$$s_j t^{n-j} - s_n = -\frac{\psi_n^2 \mathcal{E}_n^j}{1 + \mathcal{E}_n^j\phi_n} \in H^\infty, \quad (4.18)$$

where \mathcal{E}_n^j are defined as in (4.8)–(4.12) or (equivalently) by (4.13).

Proof. Apply (4.6) to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. □

5. MODEL SPACE AND TRANSFORMATION OPERATOR

Let $\mathfrak{A} \in \mathbf{Sz}$, (ϕ, ψ) be the corresponding γ -generating pair.

Definition 5.1. We define the *Faddeev–Marchenko space* M_ϕ as the Hilbert space of analytic vector-functions

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad F_1 = F_+/\psi, \quad F_2 = F_-/\bar{\psi}, \quad F_\pm \in H_\pm^2$$

with the inner product

$$\left\langle \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \right\rangle_{M_\phi} = \int_{\mathbb{T}} \begin{bmatrix} \overline{G_1} & \overline{G_2} \end{bmatrix} \begin{bmatrix} 1 & \bar{s}_0 \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} m(dt), \quad (5.1)$$

where $s_0 = -\frac{\psi}{\bar{\psi}}\bar{\phi}$.

We mention that M_ϕ comes up as a functional model space for the CMV matrix \mathfrak{A} . More specifically, we can start with the de Branges–Rovnyak model space K_ϕ : $F_\pm \in H_\pm^2$,

$$\left\| \begin{bmatrix} F_+ \\ F_- \end{bmatrix} \right\|_{K_\phi}^2 = \int_{\mathbb{T}} \begin{bmatrix} \overline{F_+(t)} & \overline{F_-(t)} \end{bmatrix} \begin{bmatrix} 1 & \phi(t) \\ \phi(t) & 1 \end{bmatrix}^{[-1]} \begin{bmatrix} F_+(t) \\ F_-(t) \end{bmatrix} m(dt)$$

and transform it as follows

$$\begin{aligned} &= \int_{\mathbb{T}} \begin{bmatrix} \overline{F_+(t)} & \overline{F_-(t)} \end{bmatrix} \begin{bmatrix} 1 & -\phi(t) \\ -\phi(t) & 1 \end{bmatrix} \begin{bmatrix} F_+(t) \\ F_-(t) \end{bmatrix} \frac{m(dt)}{1 - |\phi(t)|^2} \\ &= \int_{\mathbb{T}} \begin{bmatrix} \overline{F_+(t)} & \overline{F_-(t)} \end{bmatrix} \begin{bmatrix} 1 & -\phi(t) \\ -\phi(t) & 1 \end{bmatrix} \begin{bmatrix} F_+(t) \\ F_-(t) \end{bmatrix} \frac{m(dt)}{|\psi(t)|^2} \\ &= \int_{\mathbb{T}} \begin{bmatrix} \overline{F_+/\psi} & \overline{F_-/\psi} \end{bmatrix} \begin{bmatrix} 1 & \overline{s_0} \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} F_+/\psi \\ F_-/\psi \end{bmatrix} m(dt). \end{aligned}$$

Proposition 5.2. *Linear manifold $\begin{bmatrix} H_+^2 \\ H_-^2 \end{bmatrix}$ is contained in M_ϕ , and*

$$\left\langle \begin{bmatrix} h_+ \\ h_- \end{bmatrix}, \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \right\rangle_{M_\phi} = \left\langle \begin{bmatrix} I & \mathcal{H}^* \\ \mathcal{H} & I \end{bmatrix} \begin{bmatrix} h_+ \\ h_- \end{bmatrix}, \begin{bmatrix} g_+ \\ g_- \end{bmatrix} \right\rangle_{L^2}. \quad (5.2)$$

Proof. Let $h_\pm \in H_\pm^2$. Then

$$h_+ = \frac{\psi h_+}{\psi}, \quad h_- = \frac{\overline{\psi} h_-}{\overline{\psi}}$$

and

$$\begin{aligned} \left\| \begin{bmatrix} h_+ \\ h_- \end{bmatrix} \right\|_{M_\phi}^2 &= \int_{\mathbb{T}} (|h_+|^2 + |h_-|^2 + s_0 h_+ \overline{h_-} + \overline{s_0 h_+ h_-}) m(dt) \\ &= \|h_+\|^2 + \|h_-\|^2 + \langle s_0 h_+, h_- \rangle + \overline{\langle s_0 h_+, h_- \rangle}. \end{aligned}$$

But $\langle s_0 h_+, h_- \rangle = \langle P_- s_0 h_+, h_- \rangle = \langle \mathcal{H} h_+, h_- \rangle$, so

$$\left\| \begin{bmatrix} h_+ \\ h_- \end{bmatrix} \right\|_{M_\phi}^2 = \|h_+\|^2 + \|h_-\|^2 + \langle \mathcal{H} h_+, h_- \rangle + \overline{\langle \mathcal{H} h_+, h_- \rangle},$$

as claimed. \square

The next theorem was proved in [16, 17, 25], see also [7].

Theorem 5.3. *ϕ is Arov-regular if and only if the set $\begin{bmatrix} H_+^2 \\ H_-^2 \end{bmatrix}$ is dense in M_ϕ .*

Let $\mathfrak{A} \in \mathbf{Sz}$, (ϕ_n, ψ_n) be the sequence of γ -generating pairs related to the Schur algorithm.

Lemma 5.4. *The vectors*

$$\mathfrak{f}_n = \begin{bmatrix} \frac{t^n}{\psi_n} \\ \frac{\bar{\phi}_n}{\bar{\psi}_n} \end{bmatrix} \quad (5.3)$$

form an orthonormal system in the Faddeev–Marchenko space M_ϕ . Let $M_{\phi,+}$ be the subspace in M_ϕ spanned by those vectors, then $M_{\phi,+}^\perp$ consists of functions with $F_1 = 0$, $F_2 \in H_-^2$.

Proof. Due to recurrence (4.3),

$$\frac{t^n}{\psi_n} = \frac{t^n h_n}{\psi}, \quad \frac{\bar{\phi}_n}{\bar{\psi}_n} = \frac{\bar{\phi}_n \bar{h}_n}{\bar{\psi}}, \quad h_n \in H^\infty.$$

Using Lemma 4.4, we first compute

$$\begin{aligned} \begin{bmatrix} 1 & \bar{s}_0 \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} \frac{t^n}{\psi_n} \\ \frac{\bar{\phi}_n}{\bar{\psi}_n} \end{bmatrix} &= \begin{bmatrix} \bar{s}_0 \frac{\bar{\phi}_n}{\bar{\psi}_n} + \frac{t^n}{\psi_n} \\ s_0 \frac{t^n}{\psi_n} + \frac{\bar{\phi}_n}{\bar{\psi}_n} \end{bmatrix} = \begin{bmatrix} t^n \left(\bar{s}_n \frac{\bar{\phi}_n}{\bar{\psi}_n} - \frac{\bar{\phi}_n \psi_n \mathcal{E}_n}{1 + \mathcal{E}_n \phi_n} + \frac{1}{\psi_n} \right) \\ \frac{s_n}{\psi_n} - \frac{\psi_n \mathcal{E}_n}{1 + \mathcal{E}_n \phi_n} + \frac{\bar{\phi}_n}{\bar{\psi}_n} \end{bmatrix} \\ &= \begin{bmatrix} t^n \frac{\bar{\psi}_n}{1 + \mathcal{E}_n \phi_n} \\ -\frac{\mathcal{E}_n \psi_n}{1 + \mathcal{E}_n \phi_n} \end{bmatrix}, \quad \mathcal{E}_n = \mathcal{E}_n^0. \end{aligned}$$

Next, we assume that $m \geq n$ and compute

$$\begin{bmatrix} \frac{\bar{t}^m}{\bar{\psi}_m} & \frac{\phi_m}{\psi_m} \end{bmatrix} \begin{bmatrix} 1 & \bar{s}_0 \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} \frac{t^n}{\psi_n} \\ \frac{\bar{\phi}_n}{\bar{\psi}_n} \end{bmatrix} = \bar{t}^{(m-n)} \frac{\bar{\psi}_n}{\bar{\psi}_m} \frac{1}{1 + \mathcal{E}_n \phi_n} - \frac{\psi_n}{\psi_m} \frac{\mathcal{E}_n \phi_m}{1 + \mathcal{E}_n \phi_n}. \quad (5.4)$$

Since $\|\mathcal{E}_n\|_\infty < 1$ and due to (4.3)

$$\frac{1}{1 + \mathcal{E}_n \phi_n} \in H^\infty, \quad \frac{\psi_n}{\psi_m} \in H^\infty.$$

Hence (5.4) belongs to L^∞ , in particular, $\mathfrak{f}_n \in M_\phi$. Now (5.4) implies

$$\langle \mathfrak{f}_n, \mathfrak{f}_m \rangle_{M_\phi} = \int_{\mathbb{T}} \bar{t}^{(m-n)} \frac{\bar{\psi}_n}{\bar{\psi}_m} \frac{1}{1 + \mathcal{E}_n \phi_n} m(dt) - \int_{\mathbb{T}} \frac{\psi_n}{\psi_m} \frac{\mathcal{E}_n \phi_m}{1 + \mathcal{E}_n \phi_n} m(dt) = \delta_{mn}.$$

The first assertion follows.

To verify the second assertion, assume that vector $\begin{bmatrix} F_+/\psi \\ F_-/\overline{\psi} \end{bmatrix}$ is orthogonal to \mathfrak{f}_n for all $n = 0, 1, \dots$. As above in (5.4)

$$\begin{bmatrix} \overline{F_+} & \overline{F_-} \\ \overline{\psi} & \overline{\psi} \end{bmatrix} \begin{bmatrix} 1 & \overline{s}_0 \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} \frac{t^n}{\psi_n} \\ \frac{\overline{\phi_n}}{\overline{\psi_n}} \end{bmatrix} = \overline{F_+} t^n \frac{\overline{\psi_n}}{\overline{\psi}} \frac{1}{1 + \overline{\mathcal{E}_n \phi_n}} - \frac{\psi_n}{\psi} \frac{\overline{F_-} \mathcal{E}_n}{1 + \mathcal{E}_n \phi_n}, \quad (5.5)$$

so

$$\begin{aligned} 0 &= \left\langle \mathfrak{f}_n, \begin{bmatrix} \frac{F_+}{\psi} \\ \frac{F_-}{\overline{\psi}} \end{bmatrix} \right\rangle_{M_\phi} = \int_{\mathbb{T}} \overline{F_+} \frac{\overline{\psi_n}}{\overline{\psi}} \frac{1}{1 + \overline{\mathcal{E}_n \phi_n}} t^n m(dt) \\ &\quad - \int_{\mathbb{T}} \frac{\psi_n}{\psi} \frac{\overline{F_-} \mathcal{E}_n}{1 + \mathcal{E}_n \phi_n} m(dt). \end{aligned} \quad (5.6)$$

The second term in the right hand side of (5.6) is zero, since $F_- \in H_-^2$, so

$$\int_{\mathbb{T}} F_+ \frac{\psi_n}{\psi} \frac{1}{1 + \mathcal{E}_n \phi_n} t^{-n} m(dt) = 0, \quad n = 0, 1, \dots \quad (5.7)$$

If for the contrary

$$F_+(t) = \sum_{j \geq q} (F_+)_j t^j, \quad (F_+)_q \neq 0,$$

$(F_+)_j$ is the j -th Fourier coefficient of F_+ , then from (5.7) with $n = q$

$$\int_{\mathbb{T}} (F_+)_q \frac{\psi_n}{\psi} \frac{1}{1 + \mathcal{E}_n \phi_n} m(dt) = 0 \Rightarrow (F_+)_q \frac{\psi_n(0)}{\psi(0)} = 0.$$

The contradiction shows that $F_+ = 0$, and

$$\int_{\mathbb{T}} \begin{bmatrix} 0 & \overline{F_2} \end{bmatrix} \begin{bmatrix} 1 & \overline{s}_0 \\ s_0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ F_2 \end{bmatrix} m(dt) = \int_{\mathbb{T}} |F_2(t)|^2 m(dt) < \infty.$$

Since F_2 is of the form $F_2 = F_-/\overline{\psi}$, $F_- \in H_-^2$, ψ is outer, then, by Smirnov maximum principle, $F_2 \in H_-^2$. The proof is complete. \square

Corollary 5.5. $\left\{ \begin{bmatrix} h_+ \\ -\mathcal{H}h_+ \end{bmatrix}, \quad h_+ \in H_+^2 \right\} \subset M_{\phi,+}.$

Proof. By Proposition 5.2 the manifold $\begin{bmatrix} H_+^2 \\ H_-^2 \end{bmatrix}$ is contained in M_ϕ . By (5.2), for all $F_2 \in H_-^2$

$$\left\langle \begin{bmatrix} h_+ \\ -\mathcal{H}h_+ \end{bmatrix}, \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \right\rangle_{M_\phi} = \left\langle \begin{bmatrix} (I - \mathcal{H}^*\mathcal{H})h_+ \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ F_2 \end{bmatrix} \right\rangle_{L^2} = 0,$$

and the result follows from the second assertion of Lemma 5.4. \square

Definition 5.6. We define a unitary operator $\tilde{\mathcal{L}}$ from $M_{\phi,+}$ onto H^2 as

$$\tilde{\mathcal{L}}\mathfrak{f}_n = t^n. \quad (5.8)$$

The transformation $\mathcal{L} : H^2 \rightarrow H^2$ is defined as

$$\mathcal{L}h_+ = \tilde{\mathcal{L}} \begin{bmatrix} h_+ \\ -\mathcal{H}h_+ \end{bmatrix}. \quad (5.9)$$

\mathcal{L} is called the *transformation operator* associated to the given sequence of Verblunsky coefficients.

Proposition 5.7. *The following equality holds true*

$$I - \mathcal{H}^*\mathcal{H} = \mathcal{L}^*\mathcal{L}. \quad (5.10)$$

Proof. This follows from the unitarity of $\tilde{\mathcal{L}}$

$$\|\mathcal{L}h_+\|_{H^2}^2 = \|\tilde{\mathcal{L}} \begin{bmatrix} h_+ \\ -\mathcal{H}h_+ \end{bmatrix}\|_{H^2}^2 = \left\| \begin{bmatrix} h_+ \\ -\mathcal{H}h_+ \end{bmatrix} \right\|_{M_\phi}^2 = \langle (I - \mathcal{H}^*\mathcal{H})h_+, h_+ \rangle.$$

\square

Equality (5.10) is called the *Gelfand–Levitan–Marchenko (GLM) equation*.

Remark 5.8. Similar to Lemma 5.4 we can show that the system of vectors

$$\mathfrak{e}_{2n} = \begin{bmatrix} t^n \frac{1}{\psi_{2n}} \\ \overline{t^n \frac{\phi_{2n}}{\psi_{2n}}} \end{bmatrix}, \quad \mathfrak{e}_{2n+1} = \begin{bmatrix} t^n \frac{\phi_{2n+1}}{\psi_{2n+1}} \\ \overline{t^{n+1} \frac{1}{\psi_{2n+1}}} \end{bmatrix}, \quad n \geq 0 \quad (5.11)$$

forms an orthonormal basis for M_ϕ . Similar to Definition 5.6 we can define transformation $\tilde{\mathcal{M}}$

$$\tilde{\mathcal{M}}\mathfrak{e}_{2n} = \begin{bmatrix} t^n \\ 0 \end{bmatrix}, \quad \tilde{\mathcal{M}}\mathfrak{e}_{2n+1} = \begin{bmatrix} 0 \\ \overline{t^{n+1}} \end{bmatrix}. \quad (5.12)$$

$\tilde{\mathcal{M}}$ transforms the basis (5.11), associated to the given CMV matrix \mathfrak{A} , into the basis associated to the simplest CMV matrix (the one with $\phi = 0$, $\alpha_{-1} = -1$). Operator $\tilde{\mathcal{M}}$ is called the *transformation operator associated to the CMV matrix \mathfrak{A}* .

The transformation

$$\mathcal{M} : \begin{bmatrix} H_+^2 \\ H_-^2 \end{bmatrix} \rightarrow \begin{bmatrix} H_+^2 \\ H_-^2 \end{bmatrix}$$

is defined as a restriction of $\widetilde{\mathcal{M}}$

$$\mathcal{M} = \widetilde{\mathcal{M}} \left| \begin{bmatrix} H_+^2 \\ H_-^2 \end{bmatrix} \right. \quad (5.13)$$

Similar to (5.10) we can get

$$\begin{bmatrix} I & \mathcal{H}^* \\ \mathcal{H} & I \end{bmatrix} = \mathcal{M}^* \mathcal{M}.$$

However, it is more convenient for our purposes to use the operator $\widetilde{\mathcal{L}}$ rather than $\widetilde{\mathcal{M}}$.

Proposition 5.9. \mathcal{L} is a contraction. Matrix of \mathcal{L} with respect to the basis $\{t^k\}_{k \geq 0}$

$$\mathcal{L} = \|\mathcal{L}_{nm}\|_{n,m \geq 0}, \quad \mathcal{L}_{nm} = \langle \mathcal{L}t^m, t^n \rangle$$

is lower triangular.

Proof. The first assertion is straightforward from (5.10). For the second one we show that $\begin{bmatrix} t^n \\ -\mathcal{H}t^n \end{bmatrix}$ is in the span of $\{\mathfrak{f}_k\}_{k \geq n}$. Indeed, by using the formulae of Lemma 5.4 we get the following expression for the entries of \mathcal{L}

$$\begin{aligned} \mathcal{L}_{nm} &= \langle \mathcal{L}t^m, t^n \rangle = \left\langle \widetilde{\mathcal{L}} \begin{bmatrix} t^m \\ -\mathcal{H}t^m \end{bmatrix}, \widetilde{\mathcal{L}}\mathfrak{f}_n \right\rangle = \left\langle \begin{bmatrix} t^m \\ -\mathcal{H}t^m \end{bmatrix}, \mathfrak{f}_n \right\rangle_{M_\phi, +} \\ &= \left\langle \begin{bmatrix} t^m \\ -\mathcal{H}t^m \end{bmatrix}, \begin{bmatrix} 1 & \bar{s}_0 \\ s_0 & 1 \end{bmatrix} \mathfrak{f}_n \right\rangle_{L^2} = \left\langle \begin{bmatrix} t^m \\ -\mathcal{H}t^m \end{bmatrix}, \begin{bmatrix} t^n \frac{\overline{\psi_n}}{1 + \phi_n \mathcal{E}_n} \\ -\frac{\psi_n \mathcal{E}_n}{1 + \phi_n \mathcal{E}_n} \end{bmatrix} \right\rangle_{L^2} \\ &= \langle t^m, t^n \frac{\overline{\psi_n}}{1 + \phi_n \mathcal{E}_n} \rangle_{L^2} + \langle \mathcal{H}t^m, \frac{\psi_n \mathcal{E}_n}{1 + \phi_n \mathcal{E}_n} \rangle_{L^2} \end{aligned}$$

The last term is zero, so finally

$$\mathcal{L}_{nm} = \langle \mathcal{L}t^m, t^n \rangle = \langle \frac{\psi_n}{1 + \phi_n \mathcal{E}_n}, t^{n-m} \rangle_{L^2} = \left(\frac{\psi_n}{1 + \mathcal{E}_n \phi_n} \right)_{n-m}. \quad (5.14)$$

The latter is zero as long as $m > n$, as claimed. \square

Since $\mathcal{L}_{nn} = \psi_n(0) = \prod_{k=n}^{\infty} \rho_k > 0$, all diagonal entries of \mathcal{L} are nonzero numbers. Therefore, the matrix of \mathcal{L} has a formal inverse $\mathcal{L}^{-1} = \|\mathcal{L}_{nm}^{-1}\|$.

Theorem 5.10. The entries of the m -th column of the matrix \mathcal{L}^{-1} are the Taylor coefficients of the function $\frac{t^m}{\psi_m}$

$$\mathcal{L}_{n,m}^{-1} = \left(\frac{t^m}{\psi_m} \right)_n = \left(\frac{1}{\psi_m} \right)_{n-m}. \quad (5.15)$$

Proof. Since a product of the lower triangular matrices is a lower triangular one, need to show that for $n \geq j$

$$\sum_{m=j}^n \mathcal{L}_{nm} \left(\frac{1}{\psi_j} \right)_{m-j} = \delta_{nj}. \quad (5.16)$$

In view of (5.14)

$$\sum_{m=j}^n \mathcal{L}_{nm} \left(\frac{1}{\psi_j} \right)_{m-j} = \left(\frac{\psi_n}{\psi_j} \frac{1}{1 + \mathcal{E}_n \phi_n} \right)_{n-j}. \quad (5.17)$$

For $n = j$ (5.16) is straightforward from (5.17). For $n > j$ we turn to (4.14) and write

$$\begin{aligned} \frac{\psi_n}{\psi_j} \frac{1}{1 + \mathcal{E}_n^j \phi_n} - \frac{\psi_n}{\psi_j} \frac{1}{1 + \mathcal{E}_n \phi_n} &= \frac{\psi_n}{\psi_j} \frac{\phi_n (\mathcal{E}_n - \mathcal{E}_n^j)}{(1 + \mathcal{E}_n^j \phi_n)(1 + \mathcal{E}_n \phi_n)} \\ &= \frac{\psi_n \phi_n}{\psi_j} \frac{z^{n-j} \mathcal{P}_j}{(1 + \mathcal{E}_n^j \phi_n)(1 + \mathcal{E}_n \phi_n) \mathcal{Q}_n \mathcal{Q}_n^j} = O(z^{n-j+1}), \quad z \rightarrow 0. \end{aligned}$$

+1 comes from ϕ_n since $\phi_n(0) = 0$. Hence

$$\left(\frac{\psi_n}{\psi_j} \frac{1}{1 + \mathcal{E}_n \phi_n} \right)_{n-j} = \left(\frac{\psi_n}{\psi_j} \frac{1}{1 + \mathcal{E}_n^j \phi_n} \right)_{n-j}.$$

On the other hand, (4.16) says that the right hand side of the above equation is zero, which proves (5.17). \square

Proposition 5.11. *The system of functions $\frac{t^n}{\psi_n}$ is a Riesz basis for H^2 if and only if the matrix \mathcal{L}^{-1} defines a bounded operator on ℓ^2 , equivalently, \mathcal{L} is an isomorphism of H^2 .*

Proof. Due to the natural isomorphism between H^2 and ℓ^2 , $\frac{t^n}{\psi_n}$ is a Riesz basis for H^2 if and only if the columns of \mathcal{L}^{-1} form a Riesz basis for ℓ^2 . In turn, the columns of \mathcal{L}^{-1} form a Riesz basis for ℓ^2 if and only if both matrices \mathcal{L}^{-1} and \mathcal{L} define bounded operators on ℓ^2 . By Proposition 5.9 \mathcal{L} is always a contraction. \square

As a straightforward corollary of Theorem 5.3 and the second part of Lemma 5.4, we get

Theorem 5.12. *ϕ is regular if and only if*

$$\left\{ \begin{bmatrix} h_+ \\ -\mathcal{H}h_+ \end{bmatrix}, \quad h_+ \in H_+^2 \right\}$$

is dense in $M_{\phi,+}$.

In view of Definition 5.6 we get the following

Corollary 5.13. *The range of \mathcal{L} is dense in H^2 if and only if ϕ is regular.*

Theorem 5.14. *\mathcal{L}^{-1} is a bounded operator on H^2 if and only if ϕ is regular and $\|\mathcal{H}\| < 1$.*

Proof. By GLM equation (5.10), $\|\mathcal{H}\| < 1$ if and only if

$$\mathcal{L}^* \mathcal{L} \geq cI. \quad (5.18)$$

By Corollary 5.13, ϕ is regular if and only if the range of \mathcal{L} is dense in H^2 . The latter along with (5.18) is equivalent to the boundedness of \mathcal{L}^{-1} . \square

6. HELSON-SZEGŐ CLASS

For a function u

$$u = \sum_{k=-\infty}^{\infty} c_k t^k$$

harmonic conjugate \tilde{u} is defined as

$$\tilde{u} = -i \sum_{k=-\infty}^{\infty} \text{sign}(k) c_k t^k,$$

so the function $u + i\tilde{u}$ is “analytic”. If u is real, then so is \tilde{u} . Note that \tilde{u} does not depend on the constant Fourier coefficient u_0 . By the definition $\tilde{\tilde{u}} = -u + u_0$.

Definition 6.1. We say that w is a *positive Helson-Szegő function* if it admits a representation of the form

$$\begin{aligned} w &= C e^{u - \tilde{v}}, \quad u, v \in L^\infty \text{ (real)}, \quad \sup v - \inf v < \pi, \\ u_0 &= v_0 = 0, \quad C > 0, \end{aligned} \quad (6.1)$$

where \tilde{v} is the harmonic conjugate of v , u_0, v_0 are the constant Fourier coefficients. In this case we will say that the absolutely continuous measure $\sigma(dt) = w(t)m(dt) \in \mathbf{HS}$.

Unlike the standard convention $\|v\| < \pi/2$ we prefer to deal with

$$\sup v - \inf v < \pi$$

which is invariant under addition of any constant. Conversely, if the latter holds, then

$$\|v_c\| < \frac{\pi}{2}, \quad v_c := v + \frac{\sup v + \inf v}{2}.$$

Definition 6.2. A positive function w is said to satisfy A_2 (or Hunt–Muckenhoupt–Wheeden) condition if for all arcs $I \subset \mathbb{T}$ the following supremum is finite

$$\sup_I \langle w \rangle_I \langle w^{-1} \rangle_I < \infty, \quad \langle w \rangle_I := \frac{1}{|I|} \int_I w(t) m(dt). \quad (6.2)$$

Clearly $w \in A_2$ if and only if $1/w \in A_2$.

The following classical theorem can be found, e.g., in [22, Lecture VIII].

Theorem 6.3 (Helson–Szegő). *The following conditions are equivalent*

- (1) w is a positive Helson–Szegő function (6.1);

- (2) w satisfies the A_2 condition (6.2);
- (3) the angle is positive between $H_{+,w}^2$ and $H_{-,w}^2$ in L_w^2 :

$$|\langle g_+, g_- \rangle_w|^2 \leq \beta \|g_+\|_w^2 \cdot \|g_-\|_w^2, \quad \beta < 1.$$

Here $H_{+,w}^2$ is the closure of analytic polynomials in L_w^2 , $H_{-,w}^2$ is the closure of conjugate-analytic polynomials that vanish at the origin. It is known that for $w = |D|^2$

$$H_{+,w}^2 = D^{-1}H_+^2, \quad H_{-,w}^2 = \overline{D}^{-1}H_-^2.$$

Definition 6.4. We say that $s \in \mathbf{HS}$ if s is a canonical symbol of a Hankel operator \mathcal{H} with $\|\mathcal{H}\| < 1$.

Definition 6.5. We say that a CMV matrix \mathfrak{A} is of Helson–Szegő class ($\mathfrak{A} \in \mathbf{HS}$) if \mathcal{L}^{-1} is a bounded operator, where \mathcal{L} is the transformation operator (5.9).

In view of Theorem 5.14 $\mathfrak{A} \in \mathbf{HS}$ if and only if ϕ is regular and $\|\mathcal{H}\| < 1$. Following Arov [4], such functions ϕ are called *strongly regular*. As a consequence of the regularity of ϕ , those CMV matrices are automatically absolutely continuous. Strongly regular functions form a proper subclass of the regular ones.

The main result of this Section is

Theorem 6.6. *There is a one-to-one correspondence between \mathbf{HS} classes of CMV matrices (Verblunsky coefficients), spectral (probability) measures, and scattering functions.*

Proof. $\mathfrak{A} \in \mathbf{HS} \implies s \in \mathbf{HS}$. By Definition 6.5, $\mathfrak{A} \in \mathbf{HS}$ means that \mathcal{L}^{-1} is a bounded operator. By Theorem 5.14, the boundedness of \mathcal{L}^{-1} is equivalent to the regularity of ϕ and $\|\mathcal{H}\| < 1$. Since ϕ is regular, then, by Definition 2.9, s is canonical. Therefore, $s \in \mathbf{HS}$.

$\sigma \in \mathbf{HS} \implies s \in \mathbf{HS}$. Recall also that spectral measure in our context is always a probability measure. Hence,

$$w = \operatorname{Re} \frac{1 - \overline{\alpha}_{-1}\phi}{1 + \overline{\alpha}_{-1}\phi} = \frac{1 - |\phi|^2}{|1 + \overline{\alpha}_{-1}\phi|^2}$$

with absolutely continuous $\frac{1 - \overline{\alpha}_{-1}\phi}{1 + \overline{\alpha}_{-1}\phi}$. Assumption that w is a positive Helson–Szegő function implies that w is a Szegő function. Therefore,

$$w = \frac{|\psi|^2}{|1 + \overline{\alpha}_{-1}\phi|^2} = |D|^2, \quad \text{where } D = \frac{\psi}{1 + \overline{\alpha}_{-1}\phi}$$

In view of Theorem 6.3 (2), and since D is outer, we get that $1/D \in H^2$. Since we also have that $\frac{1 - \overline{\alpha}_{-1}\phi}{1 + \overline{\alpha}_{-1}\phi}$ is absolutely continuous, then, by Theorem 2.8 (1), ϕ is regular. Therefore, s is canonical. For $h_+ \in H^2$ and

$h_- \in H_-^2$ we have that

$$\begin{aligned} |\langle \mathcal{H}h_+, h_- \rangle| &= |\langle sh_+, h_- \rangle| = \left| \left\langle \frac{D}{\overline{D}} h_+, h_- \right\rangle \right| \\ &= \left| \left\langle \frac{1}{|D|^2} Dh_+, \overline{D}h_- \right\rangle \right| = |\langle Dh_+, \overline{D}h_- \rangle|_{w^{-1}}. \end{aligned} \quad (6.3)$$

Since $w^{-1} = |D|^{-2} \in A_2$, then, by Theorem 6.3,

$$|\langle Dh_+, \overline{D}h_- \rangle|_{w^{-1}} \leq \beta \|Dh_+\|_{w^{-1}} \|\overline{D}h_-\|_{w^{-1}} = \beta \|h_+\| \|h_-\|, \quad \beta < 1. \quad (6.4)$$

Therefore, $\|\mathcal{H}\| < 1$. Hence, $s \in \mathbf{HS}$.

$s \in \mathbf{HS} \implies \mathfrak{A} \in \mathbf{HS}$ and σ is a probability measure, $\sigma \in \mathbf{HS}$. Let s be a canonical symbol of a Hankel operator \mathcal{H} with $\|\mathcal{H}\| < 1$:

$$s = -\frac{\psi_{\mathcal{H}}}{\overline{\psi}_{\mathcal{H}}} \frac{\mathcal{E} + \overline{\phi}_{\mathcal{H}}}{1 + \phi_{\mathcal{H}}\mathcal{E}},$$

with \mathcal{E} unimodular constant and $\phi_{\mathcal{H}}$ Arov-regular. By Theorem 3.1, there exists a unique absolutely continuous CMV matrix \mathfrak{A} whose scattering function is s , moreover this \mathfrak{A} is regular. α_{-1} and the (probability) spectral density w are given by

$$\alpha_{-1} = \overline{\mathcal{E}}, \quad D = \frac{\psi_{\mathcal{H}}}{1 + \mathcal{E}\phi_{\mathcal{H}}}; \quad w = |D|^2.$$

Verblunsky coefficients of \mathfrak{A} are the Schur parameters of $\phi_{\mathcal{H}}(\zeta)/\zeta$. Since $\phi_{\mathcal{H}}$ is regular and $\|\mathcal{H}\| < 1$, then, by Theorem 5.14, \mathcal{L}^{-1} is bounded, i.e., $\mathfrak{A} \in \mathbf{HS}$.

For $h_+ \in H_+^2$ and $h_- \in H_-^2$ we have that

$$|\langle \mathcal{H}h_+, h_- \rangle| \leq \beta \|h_+\| \|h_-\|, \quad \beta < 1. \quad (6.5)$$

In view of (6.3) and by Theorem 6.3, (6.5) implies (6.4). Therefore, $|D|^2 \in A_2$, meaning that $\sigma \in \mathbf{HS}$. \square

Remark 6.7. The connection between strong regularity and A_2 condition was observed and studied by D. Arov and H. Dym in [5, 6]. They also extensively used that in their study on inverse spectral problems for canonical systems of differential equations.

Remark 6.8. Theorem 6.6 is contained in the preliminary version of the paper, see [14, Theorem 4.5, Proposition 4.7]. It was recently observed in [9, Theorem 6.3], that operator \mathcal{L} has a multiplicative structure. This observation gives a hope that the boundedness condition on \mathcal{L}^{-1} may be restated as a *constructive* condition on the Verblunsky coefficients via convergence of infinite products (series).

Definition 6.9. We say that s is a *unimodular Helson-Szegő function* if it admits a representation of the form

$$\begin{aligned} s &= ce^{i(\tilde{u}+v)}, \quad u, v \in L^\infty \text{ (real)}, \quad \sup v - \inf v < \pi, \\ u_0 &= v_0 = 0, \quad |c| = 1, \end{aligned} \quad (6.6)$$

where \tilde{u} is the harmonic conjugate of u , u_0, v_0 are the constant Fourier coefficients.

Theorem 6.10. *Canonical symbols of Hankel operators with $\|\mathcal{H}\| < 1$ are exactly unimodular Helson–Szegő functions.*

Proof. Let s be a canonical symbol of the Hankel operator \mathcal{H} with $\|\mathcal{H}\| < 1$, then, by Theorem 6.6, the unique $w \in A_2$, equivalently, w is of the form (6.1). Then $w = |D|^2$, where (taking into account our normalization $u_0 = v_0 = 0$)

$$D = D(0)e^{\frac{u+i\tilde{u}+i(v+i\tilde{v})}{2}}, \quad D(0) > 0. \quad (6.7)$$

Therefore,

$$s = -\bar{\alpha}_{-1} \frac{D}{\overline{D}} = -\bar{\alpha}_{-1} e^{i(\tilde{u}+v)}, \quad D(0) > 0,$$

and s is a canonical symbol of the Hankel operator \mathcal{H} with $\|\mathcal{H}\| < 1$.

Conversely, let s be a unimodular Helson–Szegő function, i.e., it is of the form (6.6). Then

$$s = ce^{i(\tilde{u}+v)} = c \frac{D}{\overline{D}},$$

where $|c| = 1$, D can be chosen as in (6.7). The corresponding $w = |D|^2$ is of the form (6.1). Therefore, $w \in A_2$ and, by Theorem 6.6, s is a canonical solution of the Nehari problem with $\|\mathcal{H}\| < 1$. \square

Remark 6.11. In terms of representation (6.6), the unique solution of the inverse scattering problem is given as

$$\bar{\alpha}_{-1} = -c, \quad w = Ce^{u-\tilde{v}}, \quad \int_{\mathbb{T}} w(t)m(dt) = 1.$$

7. B. GOLINSKII – I. IBRAGIMOV CLASS

Definition 7.1. A function g is in *Besov class* $B_2^{1/2}$ if

$$g = \sum_{n=-\infty}^{\infty} g_n t^n, \quad \sum_{n=-\infty}^{\infty} |n||g_n|^2 < \infty. \quad (7.1)$$

Obviously, $g \in B_2^{1/2}$ if and only if the harmonic conjugate $\tilde{g} \in B_2^{1/2}$.

Our arguments depend upon some classical results, mostly due to V. Peller [23] and S. Khrushchev and V. Peller [18]; see also [24].

Theorem 7.2. [27, Proposition 6.1.11]. *If $g \in B_2^{1/2}$, g is real, then $e^{ig} \in B_2^{1/2}$ as well.*

Theorem 7.3. [23]. *Conversely, every unimodular function s in Besov class is of the form*

$$s = t^N e^{ig}, \quad (7.2)$$

where g is real, $g \in B_2^{1/2}$, N is an integer called the index of s . N is determined uniquely, and g is up to an additive constant from $2\pi\mathbb{Z}$.

Theorem 7.4. [23]. *Every function in $B_2^{1/2}$ has a representation of the form*

$$g = g_1 + \tilde{g}_2,$$

where g_1 and g_2 are continuous functions of Besov class. If g is real, then g_1 and g_2 are also real. By means of trigonometric polynomial approximation, C -norm of g_1 or g_2 can be made as little as we want.

Theorem 7.5. [18, Corollary 1.7, p. 72]. *Let s be a unimodular function. Let $T_s = P_+ s|H^2 : H^2 \rightarrow H^2$. If*

$$\ker T_s = \ker T_s^* = \{0\}, \quad (7.3)$$

then the operators $\mathcal{H}_s^ \mathcal{H}_{\bar{s}}$ and $\mathcal{H}_{\bar{s}}^* \mathcal{H}_s$ are unitarily equivalent. The equivalence is done by the unitary factor U in the polar decomposition of T_s*

$$T_s = U \sqrt{T_s^* T_s}.$$

We start with the following

Lemma 7.6. *Let s be a unimodular function in Besov class of the index N . Then s is a unimodular Helson–Szegő function if and only if $N = 0$.*

Proof. By Theorem 7.3, $s = t^N e^{ig}$, g is real, $g \in B_2^{1/2}$. By Theorem 7.4 the function $\hat{s} = e^{ig}$ is a unimodular Helson–Szegő function (see (6.6)), so by Theorem 6.10 \hat{s} is canonical.

If $N \neq 0$, then, by Proposition 2.11, $s = t^N \hat{s}$ is not canonical, so, by Theorem 6.10, s is not a unimodular Helson–Szegő function. If $N = 0$, then $s = \hat{s}$ is a unimodular Helson–Szegő function. The proof is complete. \square

Definition 7.7. We define Golinskii – Ibragimov (**GI**) classes of CMV matrices (Verblunsky coefficients), spectral measures and scattering functions as follows

- (1) **GI** class of CMV matrices

$$\sum_{n=0}^{\infty} n |a_n|^2 < \infty, \quad \text{equivalently} \quad \prod_{n=0}^{\infty} \rho_n^n < \infty. \quad (7.4)$$

We will also write $\mathfrak{A} \in \mathbf{GI}$.

- (2) **GI** class of spectral measures consists of absolutely continuous measures with density w of the form $w = e^g$, where g is a real function in $B_2^{1/2}$. We will write $\sigma \in \mathbf{GI}$. We will also say that the spectral data $\{\sigma, \alpha_{-1}\} \in \mathbf{GI}$ if $\sigma \in \mathbf{GI}$.

- (3) **GI** class of scattering functions is the class of functions s of the form $s = e^{ig}$, where g is a real function in $B_2^{1/2}$. We will also write $s \in \mathbf{GI}$.

Lemma 7.8. *For **GI** classes of CMV matrices (Verblunsky coefficients), spectral data and scattering functions the following inclusions hold true*

$$\mathbf{GI} \subset \mathbf{HS}.$$

Proof. Inclusion **GI** \subset **HS** for spectral measures and for scattering functions follows from Theorem 7.4. To prove the inclusion for CMV matrices we show that (7.4) implies boundedness of \mathcal{L}^{-1} .

Let \mathcal{L}_m be the $m \times m$ principal block of the infinite matrix \mathcal{L} . Then the inverse matrix $(\mathcal{L}_m)^{-1}$ will be the $m \times m$ principal block of the infinite matrix \mathcal{L}^{-1}

$$(\mathcal{L}_m)^{-1} = (\mathcal{L}^{-1})_m.$$

Due to this equality, we will use the notation \mathcal{L}_m^{-1} . Note that \mathcal{L}_m is a contraction. Indeed, for ℓ_m a finite vector of length m ,

$$\|\mathcal{L}_m \ell_m\| \leq \|\mathcal{L} \ell_m\| \leq \|\ell_m\|.$$

Therefore, \mathcal{L}_m^{-1} is an expansion

$$\mathcal{L}_m^{-1*} \mathcal{L}_m^{-1} \geq I_m. \quad (7.5)$$

Now we get an upper bound on $\mathcal{L}_m^{-1*} \mathcal{L}_m^{-1}$. Due to (7.5)

$$\begin{aligned} \mathcal{L}_m^{-1*} \mathcal{L}_m^{-1} &\leq \det(\mathcal{L}_m^{-1*} \mathcal{L}_m^{-1}) I_m = |\det \mathcal{L}_m^{-1}|^2 I_m = \left(\prod_{k=0}^m \frac{1}{\psi_k(0)} \right)^2 I_m \\ &= \left(\prod_{k=0}^m \prod_{j=k}^{\infty} \frac{1}{\rho_j} \right)^2 I_m \leq \left(\prod_{k=0}^{\infty} \prod_{j=k}^{\infty} \frac{1}{\rho_j} \right)^2 I_m = \left(\prod_{j=0}^{\infty} \frac{1}{\rho_j^{j+1}} \right)^2 I_m. \end{aligned}$$

Since the bound does not depend on m , we get that the matrix \mathcal{L}^{-1} defines a bounded operator on ℓ^2 . The inclusion follows. \square

Theorem 7.9. *There is a one-to-one correspondence between **GI** classes of CMV matrices (Verblunsky coefficients), spectral data and scattering functions.*

Proof. $\sigma \in \mathbf{GI} \iff s \in \mathbf{GI}$ is straightforward. s defines σ and α_{-1} uniquely since s is canonical (see Theorem 3.1).

$\mathfrak{A} \in \mathbf{GI} \implies s \in \mathbf{GI}$. We consider $m \times m$ principal block of the GLM equation (5.10)

$$I_m - (\mathcal{H}^* \mathcal{H})_m = (\mathcal{L}^* \mathcal{L})_m \geq \mathcal{L}_m^* \mathcal{L}_m. \quad (7.6)$$

We take the determinant of the both sides to get

$$|\det \mathcal{L}_m|^2 \leq \det(I_m - (\mathcal{H}^* \mathcal{H})_m) \leq e^{-\text{tr}(\mathcal{H}^* \mathcal{H})_m}. \quad (7.7)$$

As we saw above

$$\begin{aligned} |\det \mathcal{L}_m|^2 &= \left(\prod_{k=0}^m \psi_k(0) \right)^2 \\ &= \left(\prod_{k=0}^m \prod_{j=k}^{\infty} \rho_j \right)^2 \geq \left(\prod_{k=0}^{\infty} \prod_{j=k}^{\infty} \rho_j \right)^2 = \left(\prod_{j=0}^{\infty} \rho_j^{j+1} \right)^2 > 0. \end{aligned}$$

The latter bound is independent of m . This and (7.7) imply that

$$\text{tr}(\mathcal{H}^* \mathcal{H}) < \infty. \quad (7.8)$$

The trace is computed in terms of s as

$$\mathrm{tr}(\mathcal{H}^*\mathcal{H}) = \sum_{n=-\infty}^{-1} |n| |c_n|^2,$$

where c_n are the Fourier coefficients of s . Therefore,

$$P_- s \in B_2^{1/2}. \quad (7.9)$$

We show that actually

$$s \in B_2^{1/2}.$$

We are going to apply Theorem 7.5. To this end we need to check (7.3). The kernel of T_s consists of the functions h_+ such that

$$sh_+ = h_- \in H_-^2.$$

Since $s = -\bar{\alpha}_{-1} \frac{D}{D}$, we get that

$$-\bar{\alpha}_{-1} D h_+ = \bar{D} h_-.$$

The left-hand side is in H^1 , the right-hand one is in H_-^1 . Therefore, both sides equal 0. Hence, the kernel of T_s is trivial

$$\mathrm{Ker} T_s = \{0\}. \quad (7.10)$$

The kernel of T_s^* consists of the functions h_+ such that

$$\bar{s} h_+ = h_- \in H_-^2.$$

Since s is canonical, by Proposition 2.10, this equation has only the trivial solution. Therefore, the kernel of T_s^* is trivial

$$\mathrm{Ker} T_s^* = \{0\}. \quad (7.11)$$

Due to (7.10) and (7.11) Theorem 7.5 applies and we get that $\mathcal{H}_s^* \mathcal{H}_{\bar{s}}$ and $\mathcal{H}_s^* \mathcal{H}_s$ are unitarily equivalent. Therefore, the eigenvalues of the operators $\mathcal{H}_s^* \mathcal{H}_s$ and $\mathcal{H}_s^* \mathcal{H}_{\bar{s}}$ coincide. The latter and (7.8) imply that

$$\mathrm{tr}(\mathcal{H}_s^* \mathcal{H}_{\bar{s}}) = \mathrm{tr}(\mathcal{H}_s^* \mathcal{H}_s) < \infty. \quad (7.12)$$

Hence, $P_- \bar{s} \in B_2^{1/2}$. We combine this with (7.9) to get that $s \in B_2^{1/2}$.

Since s is a unimodular Helson–Szegő function, then, by Lemma 7.6, it has the zero index.

$s \in \mathbf{GI} \implies \mathfrak{A} \in \mathbf{GI}$. By Lemma 7.8 $s \in \mathbf{GI} \implies s \in \mathbf{HS}$. Then, by Theorem 6.6, there is a unique CMV matrix \mathfrak{A} with this scattering function and the corresponding operator \mathcal{L}^{-1} is bounded. The latter allows us to rewrite GLM equation (5.10) as

$$(I - \mathcal{H}^* \mathcal{H})^{-1} = (\mathcal{L}^* \mathcal{L})^{-1} = \mathcal{L}^{-1} \mathcal{L}^{-1*}. \quad (7.13)$$

Note that the first equality in (7.13) makes sense once $\|\mathcal{H}\| < 1$, while the second does for the **HS** class only! We set

$$(I - \mathcal{H}^* \mathcal{H})^{-1} =: I + \Delta,$$

where $\Delta \geq 0$. $\text{tr}(\mathcal{H}^*\mathcal{H}) < \infty$ if and only if $\text{tr}\Delta < \infty$. Let Δ_m be $m \times m$ principal block of Δ (in the basis t^n). Then

$$I_m + \Delta_m = (\mathcal{L}^{-1}\mathcal{L}^{-1*})_m = \mathcal{L}_m^{-1}\mathcal{L}_m^{-1*}. \quad (7.14)$$

The second equality here (compare with the inequality in (7.6)) holds true since now the left factor \mathcal{L}^{-1} is lower triangular and the right factor \mathcal{L}^{-1*} is upper triangular. From (7.14) we get

$$|\det \mathcal{L}_m^{-1}|^2 = \det(I + \Delta_m).$$

Since

$$1 \leq \det(I + \Delta_m) \leq e^{\text{tr}\Delta_m} \leq e^{\text{tr}\Delta},$$

(7.4) follows. \square

Remark 7.10. As we showed in the proof of Theorem 7.9, if \mathcal{L}^{-1} is bounded, then the following version of Widom's formula holds true

$$\det(I - \mathcal{H}^*\mathcal{H}) = \prod_{j=0}^{\infty} \rho_j^{2(j+1)}.$$

For the original Widom's formula see [30], also [27, Theorem 6.2.13].

Remark 7.11. The equivalence $\mathfrak{A} \in \mathbf{GI} \iff \sigma \in \mathbf{GI}$ is the celebrated Strong Szegő Theorem (in Ibragimov's version). For the detailed exposition see [27, Chapter 6], where several independent proofs are presented. Theorem 7.9 suggests another alternate proof of this fundamental result via the scattering theory for CMV matrices.

Remark 7.12. In late 60s I. Ibragimov and V. Solev in their study of classes of Gaussian stationary processes (see [15, Chapter 4.4]) came up with the class of spectral measures of the form

$$\sigma(dt) = w(t)m(dt), \quad w(t) = |P(t)|^2 e^{h(t)}, \quad (7.15)$$

where P is a polynomial of degree N with all its zeros on the unit circle, and h is a real function from $B_2^{1/2}$. They proved that scattering functions of measures(7.15) are exactly unimodular functions s from $B_2^{1/2}$ with $\text{inds} = N$. Note that in this class solution of the inverse scattering problem is not unique. A description of the corresponding CMV matrices (similar to (7.4)) is not known.

Example 7.13. This example shows that the inclusion $\mathbf{GI} \subset \mathbf{HS}$ is proper. We consider the Jacobi weight for the unit circle

$$w(t) = C|t-1|^{2\gamma_1}|t+1|^{2\gamma_2}, \quad D(z) = C^{1/2}(1-z)^{\gamma_1}(1+z)^{\gamma_2}, \quad \gamma_{1,2} > -\frac{1}{2}$$

that enters the theory several times. First, for the choices of the parameters $\gamma_1 = 0, \gamma_2 = 2$ and $\gamma_1 = 2, \gamma_2 = 0$ we get two different weights $w_{\pm} = C|t \pm 1|^4$ with the Szegő functions $D_{\pm}(z) = C^{1/2}(1 \pm z)^2$, that have the same scattering

function $s = t^2$. Next, $w \in A_2$ if and only if $|\gamma_k| < 1/2$. Finally, the Verblunsky coefficients are known explicitly

$$a_n = -\frac{\gamma_1 - (-1)^n \gamma_2}{n + 1 + \gamma_1 + \gamma_2}, \quad n = 0, 1, \dots$$

so w is never in **GI** unless $\gamma_1 = \gamma_2 = 0$.

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